The magnetic dipole interaction in Einstein-Maxwell theory

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Abstract

I derive an exact, static, axially symmetric solution of the Einstein-Maxwell equations representing two massless magnetic dipoles, and compare it with the corresponding solution of Einstein's equations for two massless spinning particles (see gr-qc/0201094). I then obtain an exact stationary solution of the Einstein-Maxwell equations representing two massless spinning magnets in balance. The conclusion is that the spin-spin force is analogous to the force between two magnetic dipoles, but of opposite sign, and that the latter agrees with the classical value in the first approximation.

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1 Introduction

The resemblance of spin-spin interaction in general relativity and the interaction between magnetic dipoles in Maxwell's theory was pointed out by Wald [1]. Wald based his theory of the spin-spin interaction on the equations of motion for spinning test particles, and took for the magnetic interaction expressions well known in classical electromagnetism. In gr-qc/0201094, I gave an exact solution of Einstein's equations exhibiting the spin-spin interaction; here I obtain the corresponding static solution of Einstein-Maxwell theory for two magnetic dipoles and exhibit the similarity between the two solutions. Secondly, I use another solution of the Einstein-Maxwell equations to show how two massless spinning magnetic dipoles can balance. The spacetime is throughout assumed to be axially symmetric, i.e. the axes of two magnets are collinear and may be parallel or antiparallel.

In section 2 I derive the exact static solution for two magnetic dipoles, and in section 3 I write down the exact solution for two balancing spinning magnetic dipoles. Section 4 discusses the physical interpretation of the solutions, and there is a concluding section 5.

2 Exact solution for two magnetic dipoles

The field equations are those of Einstein-Maxwell theory:

$$
\begin{aligned}
R^i_k &= 2F^{ia}F_{ka} - \frac{1}{2}\delta^i_k F^{ab}F_{ab}, && (1)\\
F_{ik} &= A_{i,k} - A_{k,i}, && (2)\\
F^{ik}_{;k} &= 0, && (3)
\end{aligned}
$$

where R^i_k is the Ricci tensor, F_{ik} the electromagnetic field tensor, and A_i the vector potential; a comma denotes partial differentiation and a semi-colon covariant differentiation.

The metric may be taken in Weyl static form

$$
ds^2 = -F^{-1}[e^{\mu}(dz^2 + dr^2) + r^2 d\theta^2] + F dt^2, \tag{4}
$$

where F and μ are functions of z and r. The coordinates will be numbered

$$
x^1 = z, x^2 = r, x^3 = \theta, x^4 = t,
$$

where

$$
\infty > z > -\infty, r > 0, 2\pi \geq \theta \geq 0, \infty > t > -\infty,
$$

$\theta = 0$ and $\theta = 2\pi$ being identified. The vector potential takes the form

$$
A_i = (0, 0, \psi, 0) \tag{5}
$$

where ψ is a function of z and r.

The field equations (1)-(3) may be written in the form

$$
\begin{aligned}
R_{11} + R_{22} &= \mu_{11} + \mu_{22} - F^{-1}\nabla^2 F + \frac{3}{2}F^{-2}(F_1^2 + F_2^2) = 0, && (6)\\
R_{11} - R_{22} &= r^{-1}\mu_2 + \frac{1}{2}F^{-2}(F_1^2 - F_2^2) = 2r^{-2}F(\psi_2^2 - \psi_1^2), && (7)\\
2R_{12} &= -r^{-1}\mu_1 + F^{-2}F_1F_2 = -4r^{-2}F\psi_1\psi_2, && (8)\\
R^4_4 - R^3_3 &= e^{-\mu}[-\nabla^2 F + F^{-1}(F_1^2 + F_2^2)] = -2r^{-2}e^{-\mu}F^2(\psi_1^2 + \psi_2^2), && (9)\\
\nabla^{*2}\psi &= -F^{-1}(F_1\psi_1 + F_2\psi_2), && (10)
\end{aligned}
$$

where suffices 1 and 2 on the right mean differentiation with respect to z and r respectively, and

$$
\begin{aligned}
\nabla^2 X &= X_{11} + X_{22} + r^{-1}X_2,\\
\nabla^{*2} X &= X_{11} + X_{22} - r^{-1}X_2.
\end{aligned}
$$

Eqns (6)-(10) bear a close resemblance to (2)-(6) of [2], and from any solution of the latter set one may generate a solution of the former set by substituting

$$
f \to F^{1/2}, \quad w \to i\psi, \quad \nu \to \frac{1}{4}\mu, \tag{11}
$$

as one may verify by direct calculation. [1]

Applying this transformation to the Papapetrou solution (8)-(12) of [2] we generate a magnetic spacetime with the following metric coefficients:

$$
\begin{aligned}
F^{-1} &= \cosh^2 \xi_1, && (12)\\
i\psi &= r\xi_2, && (13)\\
\mu_1 &= 4r\xi_{11}\xi_{12}, && (14)\\
\mu_2 &= 2r[(\xi_{12})^2 - (\xi_{11})^2], && (15)\\
\nabla^2 \xi &= 0. && (16)
\end{aligned}
$$

To get a real solution of our problem we choose an imaginary solution of (16):

$$\xi = -i\left(\frac{M_1}{R_1} + \frac{M_2}{R_2}\right), \tag{17}$$

where M_1 and M_2 are moments of magnets placed on the z-axis at $z = \pm b, b > 0$ and pointing in the + or - z-direction, and $R_1 =\mid [(z-b)^2 + r^2]^{1/2} \mid$, $R_2 =\mid [(z+b)^2 + r^2]^{1/2} \mid$.

Substituting (17) into (12)-(15) we obtain

$$
\begin{aligned}
F &= \sec^2 Y, && (18)\\
\psi &= r^2\left(\frac{M_1}{R_1^3} + \frac{M_2}{R_2^3}\right), && (19)\\
\mu &= -\sum_{i=1}^{2} \frac{M_i^2 r^2 (9r^2 - 8R_i^2)}{2R_i^8} + \frac{M_1 M_2 [3(r^2+z^2-b^2)^3 + 2b^2 r^2 (9r^2 + 9z^2 - b^2)]}{2b^4 R_1^3 R_2^3} \\
&\quad +C, && (20)
\end{aligned}
$$

where C is an arbitrary constant, and where Y is given by

$$Y = \frac{M_1(z-b)}{R_1^3} + \frac{M_2(z+b)}{R_2^3}. \tag{21}$$

The physical meaning of this solution will be discussed in section 4, but I note here its resemblance to the solution for spinning particles, (15)-(17) of [2].

3 Exact solution for spinning magnetic dipoles in balance

This solution can be obtained from the Perjes-Israel-Walker (PIW) class [6] [7], which describes the stationary field of sources bearing mass, charge, magnetic moment and angular momentum. The field equations are (1)-(3). The PIW

[1] This generation procedure was first used by me in [3], though I did not describe it explicitly. An account of it was given by Kinnersley [4], and it was generalised by Fisher in [5].

solutions are free from conical singularities, but, in general they contain another singularity in the neighbourhood of the rotation axis which I have recently [8] called a torsion singularity.

A PIW solution for two particles was examined in [9], and I use it here (with a slightly different notation). To rid the solution of electric charge (which is not under consideration in this paper) one must put the parameters m_1, m_2 equal to zero; this has the effect of abolishing the masses of the particles and the torsion singularity. The result is

$$\begin{aligned} ds^2 &= -f^{-1}(dz^2 + dr^2 + r^2 d\theta^2) + f(w d\theta - dt)^2, && (22) \\ \phi &= -\epsilon f, \quad \Psi = \epsilon Y f, && (23) \end{aligned}$$

where

$$\begin{aligned} f^{-1} &= 1 + Y^2, && (24) \\ w &= 2r^2 \left(\frac{M_1}{R_1^3} + \frac{M_2}{R_2^3} \right), && (25) \\ \epsilon &= \pm 1, && (26) \end{aligned}$$

where Y is as in (21). Here ϕ is the electric potential A_4, but Ψ is the magnetic *scalar* potential. This solution will be explained in the next section.

4 Physical interpretation

In this section I discuss the exact solutions in the previous two sections, starting with the one in section 2.

We first notice that although F tends to unity and the metric becomes asymptotically flat as R_1 and R_2 tend to infinity, F becomes infinite when

$$Y := \frac{M_1(z-b)}{R_1^3} + \frac{M_2(z+b)}{R_2^3} = \pm \frac{\pi}{2}, \tag{27}$$

and one must regard this as the locus of a singularity surrounding the magnets. Our main interest in this paper is in the portion of the z-axis $-b < z < b$ where a conical singularity will enable us to deduce the force of interaction between the magnets. It is easy to see that if $M_i/b^3, i = 1, 2$ are sufficiently small there is always a part of the z-axis between the particles which is outside the singular surface (27), and the latter bifurcates into two surfaces, one around each magnet. I shall henceforth suppose this condition satisfied, and confine attention to that part of $-b < z < b, r = 0$ which is outside (27).

Using (4) and (18) we have

$$g_{44} = 1 + Y^2 + O(R^{-8}),$$

where $R^2 = z^2 + r^2$. This contains no term of order R^{-1}, so there is no term representing mass: the magnets must be considered massless. ¿From (19) we

see that the magnetic vector potential has at infinity the form expected for two magnets, parallel or anti-parallel, at $z = \pm b, r = 0$. The expression for μ can be dealt with as was ν in [2]. It represents a conical singularity either between the magnets, or outside them, according to the choice of C. Choosing the former alternative we can, as before, regard the conical singularity as a cosmic string and calculate the force which it represents. It turns out to be

$$p = \frac{3M_1 M_2}{8b^4}, \tag{28}$$

plus higher terms of order $M_1^2 M_2^2 / b^8$. Comparing this with (23) of [2] we see that the force between magnets is similar to the force between spinning particles, but opposite in sign. This agrees with Wald [1]. It is the same as that found in elementary magnetism [10]. This, incidentally, gives credibility to the cosmic string calculation of the force, which was put forward tentatively in [2].

This concludes the interpretation of the exact solution in section 3, and I turn now to the one in section 4. From the metric (22), in particular from the $d\theta dt$ term, we see that the solution refers to two particles spinning with angular momenta M_1, M_2 on the axis of symmetry at $z = \pm b$, and from (21),(23) and (24) it follows that the particles carry magnetic dipoles M_1, M_2, i.e. numerically the same as the angular momenta in relativistic units. The absence of conical singularities shows that the spin-spin and the magnetic forces balance, as expected from the above discussion of the solution in section 2. Once again the particles are massless, as one sees by examining g_{44} at infinity.

5 Conclusion

I have argued that the force between two collinear parallel or anti-parallel magnets in Einstein-Maxwell theory is given approximately by (28), which is the same as the corresponding form in classical magnetism. Of course, in the E-M case some ambiguity attaches to b because of the conical singularity between the particles.

The magnetic force has the same form as the spin-spin force found in [2], but opposite sign. This is compatible with the interpretation of the solution in section 3, in which the magnetic moments and the angular momenta are equal (in relativistic units). It also agrees with the more general approximate result of [8] in which it was shown that, in the second approximation, the forces balance if $h_1 h_2 = M_1 M_2$, h_1, h_2 being the angular momenta.

In this paper and also in [2] I have had to treat the particles as massless because appropriate exact solutions with massive particles are unknown. This means that, although spin-spin and magnetic dipole forces can be treated by analysing conical singularities, the torsion singularity [8], which arises through the coupling of mass and angular momentum, does not appear.

References

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